Applied Math 107: Graph Theory and Combinatorics Notes
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1 Lecture 1: 1/29/13

1.1 Graphs for Modeling

Let $G = (V, E)$. V is a finite set of vertices and E is a set of edges joining a pair of vertices. For example, $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 4\}\}$. We can think of a node as a point with multiple choices where an edge represents one of these possible choices.

An **undirected** graph is represented as an unordered set such as $\{1, 2\}$. A **directed** graph is represented as a directed set such as $(1, 2)$. We call a **self loop** an edge that goes from a node to itself. A **multigraph** is a graph where more than one edge exists between two nodes (a parallel edge in the same direction). Otherwise, the graph is called a **simple graph**.

The edges could represent cost or connectivity. A potential problem could involve finding a way of visiting all the nodes (traveling salesman problem) or figuring out how to visit all of the nodes exactly one and come back to the start with minimum cost. An edge could also represent a match between two nodes which could be used to match kidneys to patients or medical students to residencies.

1.2 Basic Definitions

Let $|V| = n$ and $|E| = m$.

A **walk** in $G = (V, E)$ is a sequence of vertices and edges $v_1 e_1 v_2 e_2 \ldots$ with no order restrictions.

A **path** is a walk with no vertices repeated.

Two vertices $v, w \in V$ are **connected** if there exists a walk starting at v and ending at w.

A **connected component** $G' = (V', E')$ of $G(V, E)$ is one where:

1. $V' \subseteq V$, $E' = \{(v, w) \in V'$ and $\{v, w\} \in E\}$
2. $\forall v, w \in V$, v, w are connected in G.
3. $\forall v \in V', \forall w \in V - V'$, v, w are not connected.

We say that $v, w \in E$ is **incident** to a vertex w if $u = w$ or $v = w$.

The **degree** of a vertex in G is the number of edges incident to it (self loops count twice).

Theorem: For any graph G the sum of the degrees of all the vertices is even. That is, $\Sigma_{i=1}^{n} d(v_i) = 2k$ for some k.

Proof: Each edge $\{u, v\}$ contributes to the sum of the degrees twice at both the vertex u and the vertex v. This means that $\Sigma_{i=1}^{n} d(v_i) = 2k$ which must be an even number.

Can you have a party with 7 people such that each one of them only knows 3 people? No, because $\Sigma_{i=1}^{n} d(v_i) = 3 \cdot 7 = 21$ which is not an even number.

Corollary: In any graph G the number of vertices with odd degree is even.

Proof: The sum of the degrees must be even. Then, the sum of the degrees of the vertices with even degrees must be even, so therefore to make the total sum even, some of the degrees of the vertices with odd degrees must be even. The only way this can happen is if there are an even number of odd degree vertices.

2 Lecture 2: 1/31/13

2.1 The Mountain Climber's Puzzle and its Generalization

Imagine a mountain range where M is the summit and two climbers start from different base points A and Z. Is it possible for the climbers to move along the range towards M so that they are always at the same altitude?

Answer: Yes, this is true for all mountain ranges. This has to do with the notion of a connected walk, if for a range there exists a walk from (A, Z) to (M, M) then the answer to the puzzle is yes.

Consider a graph whose vertices are pairs of points (P_L, P_R) at the same altitude where P_L is on the left side of the summit and P_R is on the right side of the summit such that one of the two points is a local minima or maxima. This set of points comprises the range graph of this mountain range. These are the points that correspond to the locations where the climbers have to make a choice – when they arrive at a peak or a valley. An edge in this graph $E = \{P_L, P_R), (P'_L, P'_R)\}$ will join these two vertices if and only if the two climbers can move constantly in the same direction (both going up or both going down) from point P_L to point P'_L and from point P_R to point P'_R.

Generalization: For any mountain range will (A, Z) be connected to (M, M)?

Claim: (A, Z) and (M, M) each have degree 1 in any range graph because A, Z are the lowest points and M is the highest point. Further, all the other nodes have degree 0,2,4.

Proof: We enumerate the 6 cases.

1. One vertex is a peak, and one is neither a peak or a valley. This means that (P_L, P_R) must have degree 2.
2. One vertex is a valley, and one is neither a peak or a valley. This means that (P_L, P_R) must have degree 2.
3. Both vertices are peaks which implies that (P_L, P_R) must have degree 4.
4. Both vertices are valleys which implies that (P_L, P_R) must have degree 4.
5. One vertex is a peak and one vertex is a valley. In order to get here, the climbers must have been going in separate directions which is impossible so (P_L, P_R) must have degree 0.
6. One vertex is a valley and one vertex is a peak. This case is exactly symmetric to the one above.

Recall the corrollary that states that for any graph the number of odd degree nodes is even. Assume that (A, Z) and (M, M) are not connected. Therefore, (A, Z) and (M, M) will be in two different connected components. Consider the connected component that contains (A, Z). This subgraph contains one vertex with degree 1. This contradicts the corrollary so (A, Z) and (M, M) must be connected.

2.2 Eulerian Walks and Hamiltonian Paths

A **walk** is a sequence $v_1 e_1 v_2 e_2 \dots e_{k-1} v_k$ where $e_i = v_{i+1}$.

A **path** is a walk with no repeated edges. A **simple path** is a walk with no repeated vertices.

A **cycle** is a walk such that $v_1 = v_k$ and no edges appear twice.

A **circuit** is a walk such that $v_1 = v_k$ and no vertices appear twice.

A **Eulerian walk or cycle** is a walk (or cycle) in which each edge appears exactly once.

A **Hamiltonian path or circuit** is a path (or circuit) that contains every vertex.

Example: The Seven Bridges of Konigsberg: In a city with seven bridges can one start at some location and find a way to go through each bridge and then come back to the starting point? This is equivalent to finding a Eulerian cycle.

Theorem: An undirected connected multigraph G has a Eulerian walk iff either none or exactly two of its vertices has odd degree. (Complete proof next lecture).

3 Lecture 3: 2/5/13

In this course, we define a path to be a walk where no vertices are repeated. A simple path will denote the situation where no edges are repeated.

Theorem: An undirected connected graph G has a Eulerian walk iff either none or exactly two vertices have odd degree.

Proof:

$\rightarrow$ Assume that a Eulerian walk exists. If this walk is a cycle then there cannot be any vertices of odd degree. If this walk is not a cycle then it must have a start and end vertex, call them u and v respectively. For any vertex $w \neq u, v$, such that w is on the walk, the walk must enter w through one edge and leave through another edge. Therefore, $deg(w)$ must be divisible by two. However, the walk begins at the start and ends and the end, so these two vertices muct have odd degree.

$\leftarrow$ Assume that G has either no vertices of odd degree or exactly two. If G has no vertices of odd degree than because it is connected, there must be a Eulerian circuit. If there are two vertices of odd degree, say u and v then there are two cases:

1. Case 1: u and v are adjacent. Create $G*$ by removing uv. Now every vertex in $G*$ has even degree so $G*$ must have a Eulerian circuit. Reorder the circuit by starting it at v and adding uv at the beginning then proceeding with the rest of the circuit. G now has a Eulerian walk.
2. Case 2: u and v are not adjacent. Create $G*$ by adding uv to G. Now every vertex has even degree so $G*$ must have a Eulerian circuit. Now we re-order the circuit so that uv is last, then when uv is removed, we have an Eulerian trail in G.

Definition: A directed graph is called **weakly connected** if when the direction of the edges is ignored a connected graph remains.

Theorem: A weakly connected and directed multigraph has a Eulerian walk iff every vertex (except possibly two) has the same indegree as outdegree. If the two exception nodes exist then one has $indegree = outdegree + 1$ and the other has $outdegree = indegree + 1$.

This problem is relevant to figuring out the optimal UPS route, or other similar problems. The strategy is to find the nodes with odd degree and then construct a matching between them (this must be possible because there is an even number of them).

Application: A Debrujn sequence for n is a sequence of 2^n symbols such that every symbol is in {0,1} so that all the 2^n cyclically continuous subsequences of n bits are distinct.

Theorem: For every positive integer n there is a Debrujn sequence. We are going to prove this by connecting the problem to a graph problem and showing that a Eulerian walk exists.

Proof: Consider $n = 4$. Construct $G = (V, E)$ such that $V = \{\alpha_1\alpha_2\alpha_3 \mid \alpha_1, \alpha_2, \alpha_3 \in \{0, 1\}\}$ and $E = \{(\alpha_1\alpha_2\alpha_3, \alpha_2\alpha_3\alpha_4) \mid \alpha_1\alpha_2\alpha_3\alpha_4 \in \{0, 1\}\}$. We see that $|V| = 8$ and $|E| = 10$. The vertices correspond to the first three symbols of the subsequence and the edges correspond to the possible subsequences (the edge connects to the last three symbols of the subsequence). For example, there is an edge connecting 000 and 001 (result of adding a 1 as the fourth symbol) and an edge connecting 000 to 000 (result of adding 0 as the last symbol). There must be a Eulerian walk in this graph because every vertex has indegree 1 and outdegree 2.

4 Lecture 4: 2/7/13

4.1 More Eulerian and Hamiltonian Walks

Example: Grey Codes: Order a 2^n bit sequence $\{0,1\}^n$ so that successive pairs differ by one bit (cycle). An example of this for $n = 3$ is $\{000, 001, 011, 111 \ldots\}$. Represent these possible $2^3 = 8$ notes as nodes and use edges to show permissable connections. We want to come up with cyclic sequence of 8 such that every pair differs by one bit. In the graph theoretical world this means that we are looking for a Hamiltonian circuit (not a Eulerian circuit because we don't want to cover every edge but rather every vertex!). In this graph we do have a Hamiltonian circuit (visible by inspection) and to generalize and prove for all n we would use induction. Figuring out whether there is a Hamiltonian circuit in a graph is NP-Complete.

Example: Assume that there are 100 people at a party and every one knows exactly k people. Can they be seated around a circular table so that everyone sits next to somesone they know? For example, if $k = 49$ the answer is no but if $k = 50$ the answer is yes.

Dirac's Theorem: If $G = (V, E)$ is undirected, and has no self loops or multiple edges, and $|V| = n \geq 3$ and every vertex has degree $\geq n/2$ then G has a Hamiltonian circuit.

Proof: By contradiction, consider $G = (V, E)$ such that $v \in V$, $d(v) \geq n/2$ and has no Hamiltonian Circuit. Add missing edges in any order until adding one more edge (any edge) creates a Hamiltonian Circuit (the graph is maximal). Let $G^* = (V, E^*)$ be this maximal graph. Consider an edge $\{u, v\} \notin E^*$ for some $u, w \in V$ then $(V, E^* \cup \{u, w\})$ has a Hamiltonian Circuit. Let this Hamiltonian circuit be $u = z\{1\}, z\{2\} \ldots z\{n-1\}, z\{n\} = w$. Suppose u is adjacent to $z\{i_1\} \ldots z\{i_k\}$ and $k \geq n/2$. Let $i_1 = 2 \leq i_2 \leq i_3 \ldots i_k$.

Punchline: Then w cannot be adjacent to $z\{i_j - 1\}$ for $1 \leq j \leq k$ for then G^* would have a Hamiltonian Circuit: $z\{1\} \ldots z\{i_j - 1\}, w = z\{n\}, z\{n-1\}, z\{n-2\} \ldots z\{i_j\}, z\{i\}$. If $k \geq n/2$ then $d(w) \leq n-k-1 < n/2$. This contradicts the assumption on G.

4.2 Matching

All sorts of marriage problems are matching problems – such as the stable matching problem where not only does everyone have to be married, but they actually have to like who they are married to. Another important problem is the problem of Distinct Representatives.

Problem of Distinct Represenatives: Pick one represenative from each committee without using the same person twice. Let the committees (R, S, T, U, V) contain the people (A, B, D, E) in the following manner: $R = (A, B, E), S = (B, D), T = (B, C, D, E), U = (A, D), V = (A, B, D)$. We can draw a compatibility graph where each committee and person is a vertex and an edge between a committee and a person implies that that person can represent that committee.

A **matching** $E' \in G = (V, E)$ is a set $E' \subseteq E$ such that if {u,v} and {w,x} are both in E' and not identical them u, v, w, x are all distinct.

A graph is **bipartite** if $V = X \cup Y$ such that $X \cap Y = \emptyset$ and $\forall \{x, y\} \in E, x \in X$ and $y \in Y$ or $y \in X$ and

$x \in Y$. $G = (X, Y, E)$ is used to represent a bipartite graph.

In a bipartite graph we say that a matching is *complete* (or saturated) if it has $|X|$ edges. (Assume without loss of generality that $|X| \leq |Y|$). If $|X| = |Y|$ then a complete matching is called a perfect matching.

In a bipartite graph, if $A \subseteq A$, $R(A)$ is $\{y \mid \{x, y\} \in E, x \in A\}$. For example, in the above committee problem, $R\{R, T\} = \{A, B.C, D, E\}$.

Hall's Theorem: $G = (X, Y, E)$ has a complete matching if and only if $\forall A \subseteq X, |R(A)| \geq |A|$.

Corollary: If $|X| = |Y| = n$, $k \geq 1$ and every node in X and Y has degree exactly k then there exists a perfect matching.

Proof: Assume otherwise. Then by Hall's Theorem there exists $A \subseteq X$ such that $|R(A)| < |A|$. But there are $k|A|$ edges incident to A and these cannot all be incident to $R(A)$ if $|R(A)| < |A|$ and the degree of all nodes is $k \geq 1$.

5 Lecture 5: 2/12/13

Hall's Theorem: $G = (X, Y, E)$ has a complete matching iff $\forall A \subseteq X$, $|R(A) \geq |A|$.

Corollary: If $|X| = |Y|$, $k \geq 1$ and every node in $X \& Y$ had degree exactly k then there exists a perfect matching.

Latin Squares: $n \times n$ where each entry is a number from $1 \dots m$ such that no repetition in a column or row exists. Can any consistent filling of some rows (some number less than n) be completed to fill a Latin square?

Example: If I have 2 rows filled out of 5, can I complete the rest so that it works?

Make a graph with a node per column and a node for possible numbers. Make edges for legal pairings. Then try to find a perfect matching. In this case, each node has degree 3, so by the corollary we can have a perfect matching.

Hall's Theorem Proof:

($\rightarrow$) If for some A, $|R(A)| < |A|$ then one cannot match the elements of A to distinct members of Y.

($\leftarrow$). Proof by induction on the number of vertices n. Inductive hypothesis: $\forall G$ with $|X| = n$ if $A \subseteq X$ and $|R(A)| > |A|$ then G has a complete matching.

Base case: $n = 1$. $\forall A$ such that $|R(A)| \geq |A|$ there is some edge incident to the only vertex in X. Therefore, G has a complete matching.

Induction Step: Assume that the inductive assumption is true for $n - 1$. Consider a G with $|X| = n$ such that $\forall A \subseteq X$. $|R(A)| \geq |A|$.

1. Case 1: $\forall A \subseteq X$, $A \neq \emptyset$ and $A \neq X$. This means that $R(|A|) > |A|$. Then, pick any $\{x, y\} \in G$ and

remove x, y and all the edges that are incident to x, y in the original graph to get a new graph G'. Then let $G' = (V', E')$. For every $A' \subseteq X'$, $R(|A|) \geq |A'|$. Since only one vertex y has been removed from Y, and $R(A')$ in G' can be at most 1 smaller than G we know that G' has a complete matching E^* from the inductive hypothesis.

2. Case 2: $\exists A_0 \subseteq A$ such that $|R(A_0)| = A_0$. Let $G' = (V', E')$ such that $V' = A_0 \cup R(A)$ and $E' = E \cap A \times R(A_0)$ (the cross product means edges that go from A to $R(A)$). Let $G'' = (V', E')$ such that $V'' = (x - A_0) \cup (Y - R(A_0))$ and $E'' = E \cap (x - A_0) \times (Y - R(A_0)$.

   Claim: G' has a complete matching

   Proof: $\forall A' \subseteq A_0$, $R(A')$ is the same as in G. Therefore, $|R(A')| \geq |A'|$ in G'. Therefore the inductive hypothesis is satisfied and by induction G' has a complete matching.

   Claim: G'' has a complete matching.

   Proof: Suppose for some $A'' \in X - A_0$, $|R(A'')| < A''$. This implies that $|R(A_0 \cup A'')| < |A'' \cup A_0|$ in G which is a contradiction. Therefore, $\forall A' \subseteq X''$, $|R(A'') \geq |A''|$. This means that G'' satisfies the inductive hypothesis. And we know that $|X''| < n$ so G'' must have a complete matching.

   Therefore, the union of complete matchings of G' and G'' is a complete matching of G.

Alternating Paths: Berge's Theorem: Assume we found a matching of size 4 between two sets of 5 nodes. Is there a larger matching? Consider the path $E \to U \to C \to S$. We flip the "in matching edges" and the "out of matching edges" to get a bigger matching (it can also be the same size).

Definition: Given a matching E_0 in $G = (V, E)$ an alternating path with respect to E_0 is a path on the edges alternating in E_0 and $E - E_0$.

Definition: A vertex u is called saturated if for some $E \in E_0$, $E = \{u, v\}$ for some v.

Definition: $S(E_0)$ is the set of all vertices saturated by E_0.

Definition: An alternating path is an **augmenting path** with respect to E_0 if it starts and ends in vertices not in $S(E_0)$.

Berge's Theorem: A matching E_0 in $G = (V, E)$ is of maximum size iff there is no augmenting path with respect to this matching E_0.

Lemma: Let $G = (V, E)$ and let E_0 and E_1 be two matchings in G. Consider $G' = (V, ((E_0 - E_1) \cup (E_1 - E_0))$. Then each connected component of G' is of one of the following kinds:

- isolated vertex
- circuit (whose edges alternate between E_0 and E_1).
- path (whose edges alternate between E_0 and E_1) and whose end points are distinct to the edges included in E_0 and E_1.

6 Lecture 6: 2/14/13

6.1 Berge's Theorem

Berge's Theorem: E_0 is a maximum matching iff there doesn't exist an augmenting path with respect to E_0.

Lemma: Let $G = (V, E)$ Let E_0, E_1 be two matchings in G. $G' = (V, (E_0 - E_1) \cup (E_1 - E_0))$. Then each of the connected components of G is either an isolated vertex, a circuit with alternating edges between E_0, E_1 or a path with alternating edges between E_0 and E_1. All of these vertices have degree 0, 1 or 2.

Proof: Proof by picture. Degree 2 vertices are incident to exactly one edge in E_1 and E_0. Degree 0 vertices are just an isolated vertices. And degree 1 vertices are incident to edge in either E_1 or E_2.

Berge's Theorem Proof:

($\rightarrow$) Assume that there exists an augmenting path with respect to E_0 by interchanging in-matching and out-of-matching edges. This means we have increased the size of the matching by 1.

($\leftarrow$) Suppose that no augmenting path with respect to E_0 exists. Let E_1 be any maximum matching in G. We know from the other direction that there exists no augmenting path with respect to E_1. Consider $G' = (V, (E_0 - E_1) \cup (E_1 - E_0))$ then G' consists of the following:

1. an isolated vertex
2. cycles alternating between edges in E_0 and E_1
3. paths:
   (a) E_0 edge at beginning and E_0 edge at end (can't happen)
   (b) E_1 edge at beginning and E_1 edge at end (can't happen)
   (c) E_1 edge at beginning and E_0 edge at end (only thing that can happen)

Only (1), (2) and (3c) are possible. In any of these situations the number of E_0 and E_1 edges are equal. $|E_0 - E_1| = |E_1 - E_0|$ which implies that $|E_0| = |E_1|$. Because E_1 was the max, E_0 must also be the max.

6.2 Maximum matching in bipartite graphs

This is the matching with the largest number of edges in $G = (X, Y, E)$.

Definition: The deficiency of $A \subseteq X$ is $B(A) = |A| - |R(A)|$.

Definition: The deficiencies of G are $B(G)$ which is the maximum of the $B(A)$ for $A \subseteq X$. Note: $B(G) \geq 0$ and $B(\emptyset) = 0$.

Generalized Hall's Theorem: In any bipartite graph $G = (X, Y, E)$ the size of the maximum matching is equal to $|X| - B(G)$.

Proof:

($\leq$) Let $A \subseteq X$ be such that $B(G) = B(A) = |A| - |R(A)|$ so $|R(A)| = |A| - B(G)$. At most $|A| - B(G)$ nodes can be matched to A. This implies that for any matching E_0, $|E_0 \leq |A| - B(G) + |X - A| = |X| - B(G)$.

($\geq$) Show/construct a matching of size $|X| - B(G)$. Add Y^* of $B(G)$ many vertices to Y and add edges $\{\{x, y\} \mid x \in X, y \in Y\}$ to get G^*. G^* satisfies the condition of Hall's Theorem so therefore G^* has a complete matching E^*. E^* has at most $B(G)$ edges going to Y^*. Therefore, removing these edges gives a matching in the original graph of size $\geq |X| - B(G)$.

6.3 Adjacent Matrix Representation

We need an efficient representation for graphs for algorithms and other things. If $(x_i, y_j) \in E$ then $B_{ij} = 1$ else $B_{ij} = 0$ in matrix B. Define a *line* as a row or a column.

Konig - Egerraty Theorem: In a $(0, 1)$ matrix the minimum number of lines that contain all the 1's is equivalent to the maximum number of 1's no two of which are on the same line which is the maximum matching.

Vertex Cover: A *vertex cover* of $G = (V, E)$ is a set $V' \subseteq V$ such that $\forall \{x, y\} \in E$ either $x \in V$ or $y \in V$.

Independent Set: An *independent set* of $G = (V, E)$ is a set $V' \subseteq V$ such that $\forall x, y \in V'$, $\{x, y\} \in E$.

Note: V' is a vertex cover, $V - V'$ is an independent set.

Konig-Egerraty Theorem Proof:

($\geq$) We want to show that the minimum of lines containing all the 1's $\geq$ the maximum number of 1's all in different rows in columns (size of maximum matching). Pick a maximum set of 1's in different lines. Any lines can contain at most one of these 1's so therefore this is true.

($\leq$) We want to show that the minimum number of lines that contains all the 1's $\leq$ size of maximum matching. The maximum matching has size $|X| - B(G)$. Let $A \subset X$ have deficiencies $B(A) = B(G)$. The number of lines containing all 1's $\leq |R(A)| + |X| - |A| = |X| - (|A| - |R(A)|) = |X| - B(G) =$ size of maximum matching, so this is true.

7 Lecture 7: 2/19/13

7.1 Finishing up matching

We present an algorithm for finding a maximum matching in a bipartite paths (Berge's Theorem) in $G = (X, Y, E)$. Start augmenting E^* at each edge and find a longer one if it exists by efficiently looking at an augmenting path.

Procedure: FINDMATCH(X,Y,E), where E' is any matching. While AUGPATHEXIST(E^*,X,Y,E) do

$E^* \leftarrow AUGMENT(E^*)$.

Procedure: AUGPATHEXIST(E^*, X, Y, E):

1. Unmark all vertices.
2. Mark all unsaturated vertices in X with p.
3. L: Mark with i all unmarked $j \in Y$ s.t. $(i, J) \in E$ and i marked by p (any i if there is a choice).
4. IF some such J is unsaturated then return YES because you have found an augmenting path.
5. Mark with p all unmarked $i \in X$ s.t. $(i, J) \in E^*$ and J is marked.
6. IF no such i is found return no augmenting matching exists ELSE go to L.

Procedure: AUGMENT(E^*): Backtrack from the saturated Y vertex found using the marks. Flip in-matching and out-matching edges.

Analysis: Call AUGPATHEXIST at most $|V|$ times. Each edge is handled at most once, so at most $|V|^2$ edges are processed, so the algorithm is cubic in $|V|^3$.

7.2 Graph Coloring

1. Given G a *vertex coloring* is an assignment of colors to vertices such that no pair of adjacent vertices gets the same color.
2. Given G an *edge coloring* is an assignment of edges to colors so that no pair of edges that share a vertex are the same color.

The **Chromatic number** of G, $\chi(G)$ is the minimum number of colors to vertex color G.

The **Chromatic index** of G, $\gamma(G)$ is the minimum number of colors to edge color G.

Example: On K_4 (complete graph), $\chi(K_4) = 4$ and $\gamma(K_4) = 3$. On K_5, $\chi(K_5) = 5$ and $\gamma(K_5) = 5$.

For an odd number of of vertices, we need more colors for the edge coloring.

The degree of G, Δ, is equvalent to the maximum degree of any vertex.

In a bipartite graph, $\chi(G) = 2$ and $\gamma(G) = \Delta$. For an arbitrary graph, $x \leq \chi(X) \leq \Delta + 1$ and $\Delta \leq \gamma(G) \leq \Delta + 1$. And for a planar graph, $2 \leq \chi(x) \leq 4$.

Example: *Map coloring:* Represent each country as a vertex. Connect vertices that share a border. We want to find the minimum number of colors of vertex color the graph.

Example: *Exam Matching:* Consider exams $A_1 \ldots A_n$ and candidates $C_1 \ldots C_m$ such that C_i takes some $B_i \subseteq \{A_1 \ldots A_n\}$. Let the vertices be the exams and let the edges be the constraints that occur when a student has two exams at the same time. We want to assign the vertices different colors so we don't have conflicts for any students.

Exam: Room Timetable: Let there be m teachers $X_1 \dots X_m$ and n groups of students $Y_1 \dots Y_m$. x_i teaches y_j for p_{ij} periods. Draw lines between X and Y. Need to edge color in a way that teacher and student groups not assigned to the same period with conflicts. Periods are equivalent to colors. The maximum number of rooms you need is the maximum number of edges that uses a color.

Theorem: $\chi(G) = 2$ iff G is bipartite iff G has no cycles of odd length.

First we show that $\chi(G) = 2 \to G$ has no cycles of odd length.

Proof: 2-color G. Then any cycle has to alternate between the two colors so it must have even length.

Next we show that if G has no odd cycles $\to \chi(G) = 2$.

Proof: Given G attempt to 2-color it as follows: Paint any node red and then colors it neighbors blue and then keep going. This means that no vertex will be able to be reached both ways (from a red and a blue node) if there does not exist in an odd cycle in the graph (path of even length + path of odd length would give a cycle of odd length).

8 Lecture 8: 2/25/13

8.1 More Coloring

Theorem: $\forall G = (V, E)$, if $degree(u) \leq \Delta$. $\forall u \in V$ then $\chi(G) \leq \Delta - 1$.

Proof sketch: Order the vertices from 1 to $n = |V|$ in any order. Color in this order coloring each vertex in a color not already used by any of its neighbors.

Brooks Theorem: $\chi(G) \leq \Delta$ if G does not contain an circuit or is not complete.

8.2 Planar Graphs

Definition; $G = (V, E)$ with no self loops or multiple edges is planar if there is an embedding of G in the plan such that no edges cross.

Theorem: Any map can be colored using at most 4 colors.

Jordan Curve Theorem: A closed non self intersecting loop divides the plane into two regions, an interior and exterior region, such that any curve joining a point in the interior to a point in the exterior must cross the original loop at some point.

Definition: For a graph G a G-configuration is any graph that can be obtained by replacing the edges of G by edges made by by inserting vertices on edges.

Theorem: (Kuratowski) A graph is planar if it contains no K_5 (complete graph on 5 nodes) or $K_{3.3}$ (bipartite graph with $|X| = |Y| = 3$ and all vertices have degree 3) graph.

Theorem: Euler's Formula: In any planar embedding of a connected planar graph the plane is divided into exactly $|E| - |V| + 2$ regions.

Proof: By induction on $|E|$. Base case: Works for $|E| = 0$ and $V = 1$. Assume that Euler's Formula holds for any connected planar graph on e edges. Now we consider a graph with $e + 1$ edges.

1. Case 1: The new edge is adjacement to only one existing vertex so we add one edge and one vertex. This does not calter the number of regions.
2. Case 2: The new edge joins two vertices. Therefore, there was already a path between these vertices. So we enclose a new region and therefore have one more region, which means Euler's formula holds.

Corollary: In a connected planar graph with $E > 1$, $|E| \leq 3|V| - 6$.

Proof: Define the degree of a region to be the number of edges traced while tracing the boundary of the region. If we sum the degrees of all the regions then we count each edge twice. So the sum of the regions is $2|E|$. Also, the degree of each region is ≥ 3, so then we plug into Euler's formula and get $3(|E| - |V| + 2) \leq 2|E| \rightarrow |E| \leq 3|V| - 6$.

Corollary: In any planar graph there is a vertex of degree ≤ 5.

Proof: Consider a graph in which every vertex has degree ≥ 6. $6|V| \leq \Sigma_{v \in V} deg(v) = 2|E| \rightarrow 3|V| \leq |E| \rightarrow 3|V| - 6 \ngeq |E|$.

Any planar graph can be colored with 5 colors by the following algorithm.

Algorithm: Given a planar graph consisting G.

1. Find a vertex V of degree ≤ 5 and recursively color $G - v$.
2. If: v's neighbors use ≤ 4 distinct colors, we color v with one of the remaining colors.
3. Else: v has 5 distinctly coloed neighbors. Number the colors in clockwise order 1 to 5.
4. Find the set of vertices than can be reached from the color 1 neighbor by paths colored as 1-3-1...
5. If the color 3 neighbor of v is not in this set then swap the color 1 and color 3 of the vertices in this set and color v 1.
6. If the color 3 neighbor is reachable by a 1,3,1.. path then find a set of vertices that can be reached by paths colored by 2,4,2,4. Swap the colors 2 and 4 and color v with 2.

9 Lecture 9: 2/28/13

9.1 Wrapping up Coloring

Theorem: The algorithm from last lecture finds a 5-coloring given a planar graph embedding.

Proof: By induction on the number of vertices. Inductive Hypothesis: The algorithm correctly colors only planar graphs on v vertices. Base case: The empty graph which is trivial. Inductive Step: The inductive hypothesis gives us:

1. Claim 1: The swaps in steps 5 and 6 map one proper coloring of $G - v$ to another. Examine neighbors of color 1 vertices, the color 3 nodes get color 1, the color 1 vertex gets color 3, all other vertices are untouched. The neighbors of color 3 vertices are done similarly, so colors 2 and 4 are swapped.

2. Claim 2: If there is a 1-3-1-3... colored path from the color 1 neighbors of v to the color 3 number then there is no 2,4,2... colored path from the color of v to the color 4 neighbor.

   Proof: The path from v to 1 to 2 and back to v is colored 1,3,1,3 and forms a closed loop. By the Jordan Curve Theorem, any path from color 2 neighbor to color 4 neighbor crosses this curve. A planar embedding of G means that edges can't cross. However, here edges must cross at a vertex colored 1 or 3. So, therefore the path is not colored 2,4,2,4.....

Definition: G is regular if every vertex has the same degree.

Lemma: A regular bipartite graph has a perfect matching

Proof: The number of edges incident to any $A \subseteq X$ is $K|A|$. Therefore, $|R(A)| \geq |A|$ since all $k|A|$ edges go to $R(A)$ and and every vertex in $R(A)$ has degree k.

Theorem: If G is a bipartite multigraph of maximum degree Δ then $\gamma(g) = \Delta$.

Proof: Add vertices and edges to G to make it regular of degree Δ. (Check that this is always possible by making the number of vertices on each side equal and then adding edges). By the Lemma there exists a perfect matching in the new graph. Color the matching edges color 1. Remove these edges to get a regular graph of degree $\Delta - 1$ and repeat.

Theorem: $G = (X, Y, E)$ is bipartite and has maximum degree Δ $\forall p \geq \Delta$. $\exists p$ disjoint matchings $M_1 \dots M_p$ such that $E = M_1 \cup M_1 \cup \dots M_p$ and $\forall i,\ .\ 1 \leq i \leq p$, $\lfloor \frac{|E|}{p} \rfloor \leq |M_i| \leq \lceil \frac{|E|}{p} \rceil$.

9.2 Network Algorithms

Shortest Paths: Consider a directed graph $G = (V, E)$, $V = \{1 \dots n\}$ and each edge (i, j) has a weight $d_{ij} > 0$ ($d_{ij} = \infty$) means that the edge $d_{ij} \notin E$. Let D_j be the total length of the minimum cost path from i to j. [$D_{ij} = \infty$ means that no path from i to j exists].

Acyclic Case: Directed graph with no directed cycles.

Fact: Can always order vertices such that for $\forall$ edge (i, j) $i < j$.

Topological Sort: Repeatedly find the vertex of $indegree = 0$, number it, and remove it from the graph.

Algorithm to find the shortest path from 1 to n

1. Topologically sort G.
2. Label vertices from left to right with minumum distance from source ($D_{ij} = 0$). For $i = 1$ to n do $D_{ij} = min_{1 \leq j1}(D_{1i} + d_{ji})$.

Analysis: n^2 steps to get D_{ij} for $1 \leq i \leq n$. n^3 steps to get D_{ki} for $1 \leq k \leq i \leq n$. Can be viewed as dynamic programming.

Cyclic Case: It is a generalization of the above.

Floyd's Algorithm: Initially $D_{ij} = d_{ij}$ [$D_{ij} = \infty$ if $(i, j) \notin E$].

1. For $k = 1$ to n do:
2. For $i = 1$ to n do:
3. For $j = 1$ to n do:
4. If $D_{ik} + D_{kj} < D_{ij}$ then $D_{ij} = D_{ik} + D_{kj}$.

10 Lecture 10: 3/5/13

The cyclic case for Floyd's Algorithm: Initially $D_{ij} = d_{ij}$, if $\{if(i, j) \notin E, d_{ij} = \infty\}$.

1. For $k = 1$ to n do:
2. Loop: For $i = 1$ to n do:
3. For $j = 1$ to n do:
4. If $D_{ik} + D_{ij}$ then $D_{ij} = D_{ik} + D_{kj}$.

Proof of correctness: Induction on the number of steps in the outer loop. The induction hypothesis is that D_{ij} is the shortest path from i to j that contains any vertices from $\{1 \ldots l\}$ initially. Base case: $l = 0$. This holds. Assume that the Inductive Hypothesis is true for $l - 1$. Now if the shortest path from i to j goes through l then it goes through l only once since $d > 0$. Therefore, in the $k = l$ loop, $D_{ik} + D_{kj} < D_{ij}$ will be found to be true since D_{il} and D_{lj} was already the shortest path by the induction hypothesis with vertices internal to $\{l \ldots l - 1\}$. As in, D_{ij} will be correctly updated. In the other case, the shortest path does not go through the node l, so no update is required.

Single Source Shortest Path Problems: Given a designated "home" node find the shortest path to each vertex from "home".

Dijstra's Algorithm: Given an undirected $G = (V, E)$ with distances $\{d_e > 0\}$. $\forall e \in E$ and designate $V_{home} \in V$. Initialize $S = (\emptyset, V_{home})$ such that $\hat{D}_{home,home} = 0$ such that $\forall (u, v) \neq (home, home)$, $\hat{D}_{u,v} = \infty$. Maintain a set of unexplored edges U incident to S in G such that $U = \{\{home, u\} \in E\}$. For $i = 1 \dots v - 1$ while the $\{u, v\} \in U$ with the minimum $\hat{D}_{home,u} + d_{uv}$ has both end points in S mark $\{u, v\}$ and delete it from U. Given $\{u, v\} \in U$ with one endpoint outside S and that $\hat{D}_{home,u} + d_{uv}$ is minimal, add $\{u, v\}$ and v into S. Mark $\{u, v\}$ and remove it from U. $\hat{D}_{home,v} = \hat{D}_{home,u} + d_{uv}$. For each neighbor u of v not in S add $\{v, u\}$ to U.

Theorem:

1. For each $u \in V$, $\hat{D}_{home,u} = D_{home,u}$
2. By tracing the edges in S from any vertex u in reverse in the Shortest Path to *home* is constructed.

Analysis: Proof by induction on the ith closest vertex to *home*. The induction hypothesis is that on iteration i, S contains some i closest vertices to *home*. $\hat{D}_{home,u}$ are the shortest path lengths for $u \in S$ and the directed paths in S are the shortest paths to *home*. Base: $S = (\emptyset, V_{home})$ such that $\hat{D}_{home,home} = 0$ which works.

Inductive Step: Consider some $(i+1)$th closest vertex v to *home*. On a shortest path to v, on its final step, it passes from some vertex u such that $D_{home,u} < D_{home,v}$. Since v is a $(i+1)$th closest vertex and u is closer to by induction hypothesis, $u \in S$ and therefore $\hat{D}_{home,u} = D_{home,u}$ and there is a shortest path to u on the edges in S. For any other edge $\{u, v\} \in U$, since v is the $(i + 1)$th closest vertex $\hat{D}_{home,u} d_{uv} = D_{home,v} \leq D_{home,v'} \leq \hat{D}_{home,u} + d_{u,v'}$. So $\{u, v\}$ is a legal choice and the algorithm chooses some equally close vertex on step i. And the shortest path to v goes through the shortest path to u and takes $\{u, v\}$.

10.1 Minimum Spanning Tree Problems

Given an undirected $G = (V, E)$ undirected with distances $d_e > 0$. $\forall e \in E$, find a spanning tree T such that $\Sigma_{e \in T} d_e$ is minimal.

Definition: A subgraph with no cycles is a forest. A connected subgraph with no cycles ia a tree. A connected subgraph including all vertices is spanning.

Claim: If $\forall e \in E$, $d_e > 0$ then a minimum spanning tree subgraph is a spanning tree.

Proof: Suppose the subgraph has a cycle. Then, we can delete one edge on the cycle and its endpoints are still connected by the path formed by the rest of the cycle. As $d_e > 0$, this new subgraph has strictly smaller total weight, so the original graph could not have been minimal.

11 Lecture 11: 3/7/13

11.1 Minimal Spanning Trees

Prim's Algorithm: Initialize $U = \{\{u, v\} \in E\}$ for a fixed $u \in V$, and $S = \{\emptyset, u\}$.

1. For $i = 1 \ldots |V| + 1$
2. WHILE the $\{u, v\} \in U$ with d_{uv} minimal has both end points in S, remove $\{u, v\}$ from U and mark $\{u, v\}$.
3. Given $\{u, v\} \in U$ with one endpoint in S and one outside and d_{uv} minimal, add $\{u, v\}$ and v to S. Mark $\{u, v\}$ and remove from S.
4. For the neighbors u' of v not in S add $\{u', v\}$ to Y.

Both Prim's and Dijstra's are Greedy Algorithms.

Claim: For any G, $d_e \in \mathbb{R}$, Prim's Algorithm returns a tree.

Proof: By induction on step i. Induction Hypothesis: There exists a unique path between any pair of nodes in S. Base case: $S = \{\emptyset, u\}$. Inductive Step: The algorithm joins some vertex $u \in S$ to a vertex $v \notin S$. Clearly this does not create a cycle as there was no prior edge to v in S_{i-1}. And by the induction hypothesis, S_{i-1} has no cycles. The unique paths to/from v in S go via the unique path to u and cross $\{u, v\}$.

Claim: In a tree there exists a unique path between each pair of vertices.

Proof: A tree is connected so it must have a path. Suppose there exists more than one distinct path. Then we can consider first point u^* where the paths diverge and the next point (after u^*) at which they meet again v^*. The two paths between u^* and v^* form a cycle so therefore the graph is not a tree which is a contradiction.

Claim: A spanning tree on V vertices has exactly $|V| - 1$ edges.

Proof: Suppose we mark a connected component of the tree starting from an arbitrary vertex one edge at a time. The tree must have $|V| - 1$ edges since after marking E edges we mark at most E new vertices (one endpoint is always in the marking component). So to mark $|V|$ verties we need $E + 1 \geq |V|$. The tree has at most $|V| + 1$ edges since each edge marked must mark a new vertex - there is already a path between any two vertices already marked, so a new edge between them would create a cycle. Therfore $E \leq |V| - 1$ and we see that $E = |V| - 1$.

Corrolary: Adding an edge to a tree creates a cycle.

Claim: The cycle created by adding an edge is unique; deleting any edge on that cycle creates a tree.

Proof: Graph is still connected and has no cycles.

Proof of Prim's Algorithm: Suppose the tree T constructed by Prim's algorithm is not a minimal spanning tree. Let e^* be the first edge that is chosen such that for the tree T^* constructed up to that point

$T^* \cup e^*$ is not consistent with any minimal spanning tree. T^* is consistent with some minimal spanning tree T'. Consider $T' \cup e^*$, this creates a cyle with at least one point in T^* and outside (as e^* crossed this boundary). There must be another edge e' on the cycle with one endpoint in T^* and one outside. Deleting e from $T' \cup e^*$ creates a new spanning tree. Moreover, since Prim's algorithm consideres e' among all of the edges with one endpoint in T^* and one outside, $d_{e'} > d_{e^*}$ (by choice of e^*). So, $T' \cup e^* - e$ is another minimum spanning tree which is a contradiction to the choice of e^* as the first inconsistent edge.

Kruskal's Algorithm: $T = \emptyset$, $\{forest\}$. Consider the edge $e \in E$ with minimum length. If it creates a cycle in T, then throw it away, else add to T. Continue until $T = |V| - 1$.

11.2 Counting

Elementary Counting: Solutions to many counting problems can be derived by the following two rules. Assume event A can oocur in m ways and event B can occur n ways.

1. A and B can happen nm ways.
2. A or B can happen $n + m$ ways.

For example, consider 26 letters, and 10 digits. The number of ways of choosing a sequence of 2 letters is 26^2. The number of ways of choosing a letter or a digit is $26 + 10$. If there are 5 Latin books, 7 Greek books and 10 French books, the number of ways to choose two books of different languages is $5 * 7 + 5 * 10 + 10 * 7$. Consider sending k kinds of postcards to n friends. There are k^n ways of sending the postcards. How about distinct postcards? This comes out to n^k ways.

12 Lecture 12: 3/12/13

Permutations: The number of ways of ordering things, for n elements you get $n!$.

Arrangements: The number of ways of choosing r out of n and ordering them. We get $n \cdot (n-1) \cdots (n - r + 1) = \frac{n!}{(n-r)!} = P(n, r)$. '

Combinations/Selections: The number of ways of selecting r out of n if order does not matter. We get $\frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!} = C(n, r) = \binom{n}{r} = \binom{n}{n-r}$.

Examples: There are 4! ways to arrange the letters of the word MATH and, and $P(4, 2) = \frac{4!}{2!} = 2!$ ways of arranging two letters. There are $C(4, 2) = \frac{4!}{2!}2! = 6$ ways of choosing two letters. The number of 8 digit binary sequences with three 0's can be written $\binom{8}{3} = \frac{8!}{5!3!} = 56$. The number of ways of seating 12 people around a table such that only relative left and right positions matter is 11!. If we don't care about left/right then it is 11!.

The Birthday Paradox: Given 15 people what is the probability that two show up with the same birthday? All the ways that this could happen is $P(365, 15) = \frac{365!}{350!}$. The probability that some share a birthday is $1 - \frac{\frac{365!}{350!}}{365^15}$.

Ramsey Theory: In any party of 6 people, either some 3 all know each other or some three all do not know each other. In a graph $G = (V, E)$ an l-clique is a set of l nodes with edges between every pair ($V' \subseteq V$ such that $\forall i, j \in V', (i, j) \in E$). Here, V' is a clique. A k independent set is a set of k nodes with no edges between any pair ($V; \subseteq V$ such that $(i, j) \in V', (i, j) \notin E$).

Proof: Any 6 node graph either has a 3-clique or a 3-independent set. Assume S is the set of people that A knows. If A knows ≥ 3 people then S ends up being a 3-indpendent set. Otherwise, the two adjacent nodes in S and A form a 3-clique.

Definition: Let $r(k, l)$ be the smallest number such that every graph of $r(k, l)$ vertices contains either a k-clique or a l-independent set. We see that $r(3, 3) = 6$.

Ramsey Theorem: $\forall k, l, r(k, l)$ is finite. The bound is $2^{k/2} \leq r(k, k) \leq 2^{2k}$.

Other Ramsey-like Theorems: Shu's Theorem states that if $\{1, 2, \dots\}$ is finitely ordered then $\exists x, y, z$ of some color such that $x + y = z$ (this is essentially partitioning into finite partitions). Vander Waerden's Theorem states that if $\{1, 2, \dots\}$ are 2-colored then at least one color class contains an arbitrary long arithmetic progression such as $x, x + y, x + 2y, \dots$.

Fact: $r(k, l) = r(l, k)$.

We want to show that $r(4, 3) < 10$. We see that $r(3, 4) \geq 9$ by drawing a pretty picture.

1. Case 1: A knows ≥ 6 people. If S contains a 3-clique then this clique plus A gives us a 4-clique. Otherwise, S contains a 3-independent set, so $r(3, 3) = 6$.
2. Case 2: A knows < 5 people. Therefore, A does not know a set T of 4 people and T is a 4-clique. Otherwise, you can find a 3-independent set.

13 Lecture 13: 3/26/13

13.1 More Ramsey Stuff

Definition: Let $r(k, l)$ be the smallest number such that every graph on $r(k, l)$ vertices containes either a k-clique or an l-independent set.

Ramsey Theorem: $\forall k, l, r(k, L)$ is finite. $2^{k/2} < r(k, l) < 2^{2k}$.

Theorem: $\forall k, l \ . \ R(k, l) = \binom{k+l-2}{k-1}$

Proof: By induction on $k + l$. Base: If $k = 2$ or $l = 2$ $R(2, l) = \binom{l}{1} = l$ or $R(k, 2) = \binom{k}{k-1} = k$ since $r(2, l) = l$ and $r(k, 2) = k$. Induction Step: For any k, l show that $R(k, l) = R(k - 1, l) + R(k, l - 1)$ works as an upper bound for $r(k, l)$ is $R(k - 1, l)$ and $R(k, l - 1)$ work then graph has total number of vertices $R(k - 1, l) + R(k, l - 1)$.

1. Case 1: A knows $\geq R(k - 1, l)$ people. So either within S we have a $k - 1$ clique including A or within S we have a $l-$ independent set.

2. Case 2: A knows $< R(k-1,l)$ people. A fails to know set of $R(k,l-1)$ people. Either T has a $k-$clique or T has a $(l-1)-$independent set including A.

Conclusion: The recurrence is $R(k,l) = R(k-1,l) + R(k,l-1)$ which is an upper bound on $r(k,l)$ and $R(k,2) = k$ and $R(2,l) = l$.

Fact: $R(k,l) = \binom{k+l-2}{k-1}$

Proof: By induction on $k+l$, $R(k,l-1) + R(k-1,l) = R(k,l)$ we get $\binom{k+l-3}{k-1} + \binom{k+l-3}{k-2} = \binom{k+1-2}{k-1}$

Corollary: $r(k,k) \leq R(k,k) = \binom{2k-2}{k-1} < 2^{2k}$

Theorem: $r(k,k) \geq 2^{k/2}$

Proof: (probabilistic method) Let $G = (V,E)$ be a random graph with $|V| = n = 2^{k/2}$ and each edge exists with probability $1/2$. For any graph on k vertices the probability it is a clique is the probability that it is an independent set is $\frac{1}{2}^{\binom{k}{2}}$. That is the probabilty that a k-vertex graph is a clique or an independent set. The number of k-tuples of vertices in G is $\binom{n}{k}$ so P(there exists at least a k-clique or a k-independent set) $< \binom{n}{k}\frac{2}{2^{\binom{k}{2}}}$. Therefore, P(there exists a k-clique or a k-independent set) < 1. With non-zero probability no k-clique or no k-independent set exists in G. So, $r(k,k) > 2^{k/2}$.

13.2 Asymptototic Estimates

Example: Toss n coins. The probability of exactly $n/2$ nickels is:

$\frac{\binom{n}{n/2}}{2^n} = \frac{n!}{(n/2)!(n/2)!2^n} \approx \frac{(n/e)^n\sqrt{(2\Pi n)}}{((n/2e)^{n/2}\sqrt{\Pi n})^2 \cdot 2^n} = \sqrt{\frac{2}{\Pi}} \cdot \frac{1}{\sqrt{n}}$.

Birthday Paradox: Suppose there are n distinct days in a year. There exists $m = (1+\epsilon)nlog_e n$ people. The probabilty that a specific birthday is missed is $(1-1/n)^{(1+\epsilon)nlog_e n}$. The probability that some birthday is missed $\leq n(1-1/n)^{(1+\epsilon)nlog_e n}$ and this goes to 0.

14 Lecture 14: 3/28/13

14.1 Random Graphs

These are useful because they occur frequently in the natural world (for example social networks). We have n labeled vertices with the probability of $(i,j) \in E = p$ and we call this $G_p(n)$.

Theorem: $Prob(G_{1/2}(n)$ is connected$) \to 1$ as $n \to \infty$

Proof: $Prob(G_{1/2}(n)$ is not connected$) \leq \Sigma_{i=1}^{n/2} prob(G_{1/2}(n)$ splits into parts of size i such that $n-i$ not connected $\leq \Sigma_{i=1}^{n/2}\binom{n}{i}(1/2)^{i(n-i)}$ (this is because the edges crossing between i and $n-i$ must all not exist)

$= \Sigma_{i=1}^{n/2} \frac{n(n-1)...(n-i+1)}{i!} \frac{1}{2^{i(n-i}} \leq \Sigma_{i=1}^{n/2} \frac{n^i}{2^{i(n-i)}} \leq \Sigma_{i=1}^{n/2} \frac{n^i}{2^{in/2}}$ (this is because if $i \leq n/2$ then $n-i \geq n/2$ and therefore $(n-i)^2 \geq in/2) = \Sigma_{i=1}^{n/2} \frac{n}{2^{n/2}}^i \leq \frac{n}{2} \cdot \frac{n}{2^{n/2}} \to 0$ as $n \to \infty$.

In fact, the threshold of connectivity is $p = \frac{ln(n)}{n}$.

14.2 Proofs of Existence by Counting

Definition: A tournament is a directed graph in which between every pair of vertices there exists an edge in exactly one direction.

Theorem: There exists some tournament on n vertices with $n!2^{-n+1}$ paths.

Proof: For any Hamiltonian path p, there exist $2^{(\binom{n}{2})-(n-1)}$ tournaments that contain it. Therefore, the number of ways of choosing a pair (T,p) where T is a tournament that contains the Hamiltonian path p is $n!2^{\binom{n}{2})-(n-1)}$. If p is contained in T and there are $2^{\binom{n}{2})-(n-1)}$ edges, then some tournament exists with $\geq \frac{n!}{2^(n-1)}$ edges.

Definition: A tournament is transitive if $(i,j) \in E$ and $(j,k) \in E$ implies that $(i,k) \in E$.

Theorem: There exists some tournament on n vertices that has no transitive subtournament of size $\lceil 2log_2 n + 2\rceil$.

Proof: Let T be the set of all tournaments on n vertices. $|T| = 2^{\binom{n}{2}}$. Let T^s be the set of all tournaments on n vertices that contain some transitive subtournament on s vertices. Then $T^s = \cup_{A \subseteq \{1...n\}|A|=s}$ and $\cup_{\sigma \text{ orderings on elements from } A}$. Therefore, $T^s_{A,\sigma}$ means the set of tournaments on which a set of size A is transitive in ordering. Then $T^s_{A,\sigma} = 2^{\binom{n}{2}-\binom{s}{2}} \to |T^s| \leq \binom{n}{s} s! 2^{\binom{n}{2}-\binom{s}{2}}$ (the first two terms are to choose the vertices for s and then to order them) $\leq (\frac{n!}{(n-s)s!} \cdot s! \cdot 2^{-\binom{s}{2}}) 2^{\binom{n}{2}} = \frac{n^s}{2^{\frac{s(s-1)}{2}}} \cdot 2^{\binom{n}{2}} = \frac{n}{2^{\frac{s-1}{2}}}^s \cdot 2^{\binom{n}{2}}$.

This reduces to the proof that $\frac{n}{2^{\frac{s-1}{2}}} < 1$ if $s \geq \lceil 2log_2 n + 2\rceil$.

14.3 Summing Expectations

Let X be a random variable. Take a value x_i with probability $f(x_i)$ where $i = 1 \ldots n$.

Definition: The expectaton of X is $E(X) = \Sigma_{i=1}^n x_i f(x_i)$.

Let Y be another random variable that takes value y_i with probability $g(y_i)$.

Fact: $E(X+Y) = E(X) + E(Y)$. If there is a set of random variables $X_1 \ldots X_m$ then $E(X_1 + X_2 \ldots X_m) = \Sigma_{i=1}^m E(X_i)$

Example: Consider 10 coins. Let X_i be 1 if the ith coin lands on heads and 0 otherwise.

For all i, $E(X_i) = 1 \cdot 1/2 + 0 \cdot 1/2 = 1/2$. Therefore, $E(X_1 + X_2 + \ldots X_{10}) = 10 \cdot E(X_i) = 5$.

Proof of fact: Let $h(x_i, y_j) = Prob(X = X_i \cap Y = Y_j)$. Then, $E(X) = \Sigma_i X)$ if $(X_i) = \Sigma X_i \Sigma_j h(X_i, Y_j)$ and $E(Y) = \Sigma_j Y_j g(Y_j) = \Sigma Y_i \Sigma_i h(X_i, Y_j)$. Then, $E(X+Y) = \Sigma_i \Sigma_j (X_i + Y_j) h(X_i, Y_j) = E(X) + E(Y)$.

14.4 Coupon Collector's Problem

Consider n types of coupons and m picks with replacement. Let A_i be the probability that coupon i is never picked. This is equivalent to $(1-1/n)^m$. Let X_i be 1 if coupon i is never drawn and 0 otherwise. Then $E(X_1 + \ldots X_n) = nE(X_i) = n(1-1/n)^m$. If $m = (1+\epsilon)ln(n)$ then $nE(X_i) = n(1-1/n)^{n(1+\epsilon)ln(n)} = n(1-1/n)^n \to \frac{n}{e^{ln(n)}}^{1+\epsilon} = \frac{n}{n^(1+\epsilon} = n^{-\epsilon}$. If $\epsilon = 1/2$ then we expect to miss $\frac{1}{\sqrt{(n)}}$ components. If $\epsilon = -1/2$ then we expect to miss $\sqrt{(n)}$ components.

Fact: Given a random directed graph G with n vertices where each edge is present independently with probability $p = k/n$, $k \geq 3$:

- $Prob(G_p(n)$ has a Hamiltonian Circuit$) \to 0$ as $n \to \infty$
- $E($number of Hamiltonian Circuits in $G_p(n))$ is exponential in n.

15 Lecture 15: 4/2/13

15.1 More Random Graphs

Fact: Given a random graph $G_{k/n}(n)$ where $p = k/n$:

- The probability that G has a Hamiltonian Circuit $\to 0$ as $n \to \infty$
- $E($the number of Hamiltonian Circuits in $G)$ is exponential in n

Proof:

- Let * be the probability that there exists at least one edge directed out of vertex i. $* = 1-(1-p)^{(n-1)} \leq 1-(1-k/n^{k/n^k})$. Since, $(1-k/n)^{n/k} \to 1/e$ as $n \to \infty$, t is greater than $1/(2e)$. This means that $x \leq 1 - (1/(2e)^k) < c$ for some $c > 1$. And so the probability that G has a Hamiltonian circuit $\leq$ the probability that there is an edge from every vertex $\leq c^n$. This approaches 0 as $n \to \infty$.
- Let X be a variable that represents whether the ith permutation of a graph is a Hamiltonian Circuit (t is either 1 or 0). $E[\#HC \in G] = E[\Sigma_i X_i] = \Sigma E[X_i] = \Sigma_{i=1}^{(n-1)!}$. $E($number of HC's in $G) = E(\Sigma_i X_i) = \Sigma E(X_i) = \Sigma_1^{(n-1)!}(k/n)^n = (n-1)!(k/n)^n = 1/n \cdot n! \cdot (k/n)^n \geq 1/n \cdot (n/e)^n \cdot (k/n)^n = 1/n \cdot (k/e)^n$. The last step is by Stirling's Approximation.

15.2 Combinatorial Facts

15.2.1 Arrangements with Repetitions

Arrangements with repetition: Consider n objects of t kinds such that we want a_1 of first kind, a_2 of second kind.....We get $\frac{n!}{a_1!a_2!...a_t!}$ arrangements. For example, the number of arrangements of MATHEMATICS is 11!/(2!2!2!).

15.2.2 Distribution Problems

How many ways are there to put n distinguishable/indistinguishable objects into r distinguishable/indistinguishable bins?

Bin/Obj	Distinct	Undistinct
Dist	n^r	$\binom{r+n-1}{r}$ [stars and bars]
Undist		

Variations on this include situations where the ordering of the objects matters or where you need at least k in every bin.

1. Consider r distinct objects and n distinct bins, where the order in each bin does not matter. We get n^r.
2. Consider r distinct objects and n distinct bins, where order (within each bin) matters. We combine permutations and stars and bars. We get $r!\binom{n+r-1}{r} = \frac{r!(n+r-1)!}{r!(n-1)!} = \frac{(n+r-1)!}{(n-1)!}$.
3. Consider r indistinguishable balls and n distinguishable bins. $\binom{r+n-1}{r} = \frac{r!(n+r-1)!}{r!(n-1)!}$.
4. Consider r indistinguishable balls and n distinguishable bins but we require at least 1 object in each bin. We get $\binom{r-n+n-1}{r-n} = \binom{r-1}{r-n} = \frac{(r-1)!}{(r-n)!(n-1)!}$.
5. Consider r distinct objects and n indistinct bins with at least one object in each bin and where the ordering within the bins doesn't matter. $S(r,n)$ is Stirling's number of the second kind. It can be computed recursively, $S(n,r) = S(n-1,r-1) = nS(r-1,n)$ and $S(r,n) = 1/(n!)\Sigma_{i=0}^{n}(-1)^i\binom{n}{i}(n-1)^r$.
6. Consider r distinct objects and n indistinct bins. We get $\frac{n^r}{n!}$.
7. Putting Indistinct objects into indistinct bins is just like partitioning integers. For example $5 = 1+1+1+1+1 = 1+1+1+2 = 1+1+3 = 1+4 = 1+2+2 = 2+3$. This can be modeled with a Ferrer diagram (dots representing possible partitions) but generally it is best to handle this with a generating function.

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Example: Consider partitioning 5. $P(5,1)+P(5,2)+P(5,3)+P(5,4)+P(5,5) = 1+2+2+1+1 = 7$.

Bin/Obj	Distinct	Undistinct
Dist	n^r	$\frac{(r+n-1)!}{r!(n-1)!} = \binom{r+n-1}{n-1}$
Undist	$\Sigma_{j=1}^{n} S(r,j)$ (number of items into J non empty boxes	$\Sigma_{i=0}^{r} P(n,i)$ (number of partitions of n into i parts)

16.1 Proof of Combinatorial Identities by Stories

1. $\binom{n}{r} = \binom{n}{n-r}$: Choosing r people to be in a group out of n is equivalent to chosing $n-r$ people to be in a group.
2. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{i} \ldots = 2^n$: 2^n is the total number of subsets and $\binom{n}{i}$ is the number of subsets of n of size i.
3. $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r-1}$: Either you choose some element x and then need to choose the other $r-1$ elements out of n or you don't choose the first element first and just choose the r out of n.
4. $\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{i}\binom{m}{r-i} + \ldots + \binom{n}{r}\binom{m}{0}$: This represents all the ways of choosing the first element from n and the rest from m, then two elements from n and the rest from m and so on.
5. $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$
6. $\Sigma_{k=0}^{m} \binom{m}{k}\binom{n-k}{]} m = \Sigma_{j=0}^{min(m,n)} \binom{n}{j}\binom{m}{j} 2^j$
7. $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$: This represents the number of non-identity permutations. Let x_i be the number of permutations when x_{i+1} is not in correct place but $x_{i+2} \cdots x_{n+1}$ are. $x(i) = i \cdot i!$. i is the number of positions of the $i+1$th element and $i!$ is the permutation of the x_i.

16.2 Counting using Generating Functions

Definition: If $a_0, a_1 \cdots a_i \cdots$ is a finite or infinite sequence then its ordinary generating function is $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_i x^i + \ldots$. The exponential generating function is $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_i x^i + \ldots = 1! + 2! + \ldots i! + \ldots$.

Example: $a_0 = \binom{n}{0}, a_1 = \binom{n}{1}, \ldots$.

Binomial Theorem: Consider the Ordinary Generating Function (OGF) $= \Sigma_{i=0}^{n} x^i \binom{n}{i} = (1-x)^n$. $(1+x)(1+x)(1+x)$ has 2^3 terms and the coefficient of x^2 represents the ways of choosing two $x's$ from the three parentheses.

Example: How many ways are there of choosing a set of r red balls from a set of ∞ red balls? $OGF = 1 + x + x^2 + x^3 \ldots$.

Example: The number of ways of choosing of r red balls from a set of n red and blue balls. $OGF = 1 + 2x + 3x^2 + \ldots + 8x^7 + (i+1)x^i + \ldots = f(x) \cdot g(x)$ (represents the coefficient for x^r).

Multiplication Principle of Ordinary Generating Functions If $f(x)$ counts the number of ways of choosing from set A and $g(x)$ counts the number of ways of choosing from set B then $f(x)g(x)$ counts the number of ways of of choosing from set $A \cup B$ if the objects in A are indistinguishable from the objects in B.

Proof: If $f(x) = \Sigma_{i=0}^{n} f_i x^i$ and $g(x) = \Sigma_{i=0}^{n} g_i x^i$ then $f(x)g(x) = \Sigma_t (f_0 g_0 + f_1 g_{t-1} + \ldots f_t g_0)x^t$ by definition of multiplication for polynomials. But at the same time, the coefficient of x^t represents the number of combinations of t things picked from the mixture.

If $f(x) = 1 + x + x^2 + \ldots = \frac{1}{x-1}$ and $g(x) = 1 + x + x^2 + \ldots = \frac{1}{x-1}$, then a mix of red and blue balls is represented by $f(x) \cdot g(x) = \frac{1}{(1-x)^2}$. A mixture of 7 colors is represented by $\frac{1}{(1-x)^7}$

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17.1 More Generating Functions

Let the generating function for the red balls be $f(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x}$. Let the generating function for the blue balls be $g(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x}$. The number of ways to pick from a mix of red and blue balls is $h(x) = f(x)g(x) = \frac{1}{(1-x)^2}$. The number of ways of selecting 5 objects out of 4 types of objects can also be calculated by using a generating function. The generating function for one object is $(1+x+x^2+\ldots)$. The generating function for 4 objects is $(1 + x + x^2 + \ldots)^4$. The coefficient of x^5 represents the number of 4 integer partitions of the number 5. $x^{e_1} x^{e_2} x^{e_3} x^{e_4} = x^5$ where $e_1 + e_2 + e_3 + e_4 = 5$. This is the same as the number of ways of distributing 5 indistict objects into 4 distinct bins which is $\binom{5+4-1}{4-1} = \binom{5+4-1}{5}$.

Binomial Theorem: $(1-x)^{-n} = \Sigma_0^{\infty} = \binom{n+i-1}{i} x^i$ and $(1+x)^n = \Sigma_0^n \binom{n}{i}$. The coefficient of x^t is $\binom{n}{t}$.

Let's say that we impose the constraint that we can have at most 2 objects of each type. This means we get the generating function $(1 + x + x^2)^4$.

Example: The number of ways of choosing 3 identical green marbles is $1 + x + x^2 + x^3$. The number of ways of choosing 4 identical blue marbles is $1 + x + x^2 + x^3 + x^4$. The number of ways of choosing t items from this mixture is the coefficient of x^t in $(1 + x + x^2 + x^3)(1 + x + x^2 + x^3 + x^4)$. But if we want at least one blue and one green marble we need the coefficient of x^t in the function $(x + x^2 + x^3)(x + x^2 + x^3 + x^4)$.

Example: We want the number of ways to distribute 16 indistinct objects into 5 distinct bins such that there are at least two in each bin. We want the coefficient of x^{16} in $(x^2 + x^3 + \ldots)^5 = x^2(1 + x + x^3 + \ldots)^5$. This is is also equivalent to finding the coefficient of x^6 in the polynomial $(1+x+\ldots)^5$. This last polynomial simplifies to $\frac{1}{(1-x)^5}$ and by Binomial Theorem we see that its 6th coefficient is $\binom{5+6-1}{6} = \binom{10}{6}$.

17.2 Integer Partitions

For example, $13 = 1+1+1+2+2+2+4$. We want a power series such that the coefficient of x^i represents the number of partitions of x. The number of partitions i when each partition is of size 1 is $1+x+x^2+\ldots = \frac{1}{1-x}$. The number of partitions i when each partition is of size 2 is $1 + x^2 + x^4 + \ldots = \frac{1}{1-x^2}$. The number of ways to partition x where each partition is of size 1 or 2 is $\frac{1}{1-x} \frac{1}{1-x^2}$. So we need the coefficient of x^i in this polynomial.

General Formula: The generating function for the number of partitions is $\Pi_{i=1}^{\infty} \frac{1}{1-x^i}$. The number of partitions of the integer n is the coefficient of x^n.

The number of ways of partitioning n into m parts involves introducing a new variable t. This comes out to $(1+xt+x^2t^2+\ldots)(1+x^2t+x^4t^2+\ldots)(1+x^3t+x^6t^2+\ldots) = \frac{1}{1-xt}\frac{1}{1-x^2t}\frac{1}{1-x^3t}$. Formally, we need the coefficient of x^nt^m in the polynomial $\Pi_{i=1}^{\infty}\frac{1}{1-tx^i}$.

Example: We have 200 chairs and 4 rooms. We can have 0 or 20 or 40 or 60 or 80 or 100 chairs per room. Find the number of ways we can do this when the rooms are distinct and indistinct. If the rooms are distinct we need the coefficient of x^{200} in $(1+x^{20}+x^{40}+x^{60}+x^{80}+x^{100})^4$. If the rooms are not distinct then we need the coefficient of $x^{200}t^4$ in $(1+t+t^2+t^3+t^4)(1+x^{20}t+x^{40}t^2+x^{60}t^3+x^{80}t^4)+(1+x^{40}t+x^{80}t^2+x^{120}t^3+x^{160}t^4)+(1+x^{60}t+x^{120}t^2+x^{180}t^3)(1+x^{80}t+x^{100}t^2)+(1+x^{100}t+x^{200}t^2)$

Proving combinatorial identities using Generating Functions usually works since most generating functions have very simple expansions (details next lecture)

18 Lecture 18: 4/11/13

18.1 Proving Combinational Indentities using Generating Functions and Algebraic Manipulations

We know that:

$1+x+x^2+\ldots = \frac{1}{1-x}$

$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots = e^x$

$\Sigma_{i=0}^{n} ix^i = \frac{x}{(1-x)^2}$

Example: Take the derivative of $(\Sigma_i x^i = \frac{1}{1-x})$ and we get $\Sigma_i ix^{i-1} = \frac{1}{(1-x)^2} = \Sigma_i x^i = \frac{x}{(1-x)^2}$. Could also do this by the binomial theorem and get that $\frac{x}{(1-x)^2} = x(1+2x+\cdots+\binom{n+i-1}{i}x^i+\cdots)$ so we get that $\binom{2+i-1}{i} = i$.

Example: $\binom{n}{1}n+2\binom{n}{2}+\ldots+n\binom{n}{n} = n2^{n-1}$. This represents picking a committee of size i out of n and then choosing a leader. (The right side first chooses a leader and then counts the total number of subcommittes). This is a story proof.

Algebraic Proof: We know that $\binom{n}{1}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots\binom{n}{n}x^n = (1+x)^n$. The derivative of this is $\binom{n}{1}+2\binom{n}{2}x+\ldots+n\binom{n}{n}x^{n-1} = n(1+x)^{n-1}$. If we set $n=1$ we get that $\binom{n}{1}n+2\binom{n}{2}+\ldots+n\binom{n}{n} = n2^{n-1}$ and we are done!

Example: $\binom{n}{0}^2+\binom{n}{1}^2+\cdots+\binom{n}{n}^2 = \binom{2n}{n}$. Recall that that $\binom{n}{i}^2 = \binom{n}{i}\binom{n}{n-i}$. We know that in the polynomial $(1+x)^n(1+x)^n$ x^n has a coefficient of $\binom{n}{i}\binom{n}{n-i}$. This is the same as the coefficient of x^n on $(1+x)^{2n}$ which will be $\binom{2n}{n}$.

Example: $\binom{n}{0}+\binom{n}{2}+\cdots+\binom{n}{n} = \frac{2^n}{2}$ for even n. We know that $(1+x)^n = \Sigma_i\binom{n}{i}x^i$ and $(1-x)^n = \Sigma_i(-1)^i\binom{n}{i}x^i$.

We then sum: $(1+x)^n + (1-x)^n = 2(\binom{n}{0} + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n)$. Then set $x = 1$ and we get $2^n = 2(\binom{n}{0} + \binom{n}{2} + \ldots + \binom{n}{n})$. This comes out to $2^{n-1} = \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n}$.

18.2 Exponential Generating Functions

Definition: The OGF of a sequence $a_0, a_1, \ldots$ is $A(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + \ldots + a_i \frac{x^i}{i!} + \ldots$. This is useful for counting when order matters (permutations). $(1+x)^n = \Sigma_i \binom{n}{i} x^i = \binom{n}{0} + \ldots + \frac{n!}{r!(n-r)!}x^r$. If the order matters then the number of ways of arranging r objects out of n would be $\frac{n!}{(n-r)!} = P(n,r)$.

Definition: The Exponential Generating Function (EGF) is for picking i objects from a set of infinitely many identical objects. Because they are identical there is only one way of picking i of them. $e^x = 1 + x + \frac{x^2}{2!} + \ldots$.

If you have a mixture of colors: $(1 + x + \frac{x^2}{2!} + \ldots)^2 = e^{2x}$ where $e^{2x} = \Sigma_{i=0}^{\infty} \frac{(2x)^i}{i!} = \Sigma 2^i \frac{x^i}{i!}$.

Definitions:

$e^{nx} = 1 + nx + \frac{n^2x^2}{2!}$

$e^x - 1 - x = \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$1/2(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

$1/2(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

Multiplication principle for EGFs Let a_i be the number of ways of choosing i items from bin 1 if order matters and let b_j be the number of ways of choosing j items form bin 2 if order matters. The number of ways of choosing i from the first and j from the second is (if order matters) $\frac{(i+j)!}{i!}j!$. The number of ways of picking t things altogether is $\Sigma_{i+j=t} a_i b_j \frac{t!}{i!j!} = \Sigma_{t=0}^{\infty} \Sigma_{i=0}^{t} \frac{x^t}{i!(t-i)!} a_i b_{t-i} = \Sigma_i a_i \frac{x^i}{i!} \Sigma_j b_j \frac{x^j}{j!}$.

Conclusion: if $A(x)$ and $B(x)$ are EGFs for 2 different objects then $A(x)B(x)$ is the EGF for choosing from the mixture.

Example: Consider the number of ways of converting an r digit number to base 4 such that the digits 1,2,3 appear at least once. We need the coefficient of x^n in $(e^x - 1)^3 e^x$. $(e^x - 1)$ is the EGF of picking from a set of 1's and at least getting one. Multiplying out we get, $e^{4x} - 3e^{3x} + 3e^{2x} - e^x$. We need the coefficient of x^r. We know that $e^{nx} = \Sigma_i n^i \frac{x^i}{i!} = \Sigma_{i=0}^{\infty} \frac{x^i}{i!}(4^i - 3 \cdot 3^i + 3 \cdot 2^i - 1)$. The coefficient of x^r is $4^r - 3 \cdot 3^r + 3 \cdot 2^r - 1)$.

19 Lecture 19: 4/22/13

19.1 Recurrence Relations

Consider the specification of a sequence $a_0, a_1 \ldots a_n$ by $a_n = f(a_{n-1}, a_{n-1}, \ldots)$ and boundary conditions such as $a_0 = x$ and $a_1 = y$. We can write $a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-r})$ then specify $a_0 \ldots a_r$. We want to find the closed form expressions for the asymptotic growth.

Example: Assume you have n stairs to climb. You take 1 or 2 stairs with each step. The time it takes to go from 0 to n steps is $a_n = a_{n-1} + a_{n-2}$ where the first part of the sum is the situation where the last move was to make one step and the second part of the sum represents the situation where the last move was two steps. This is equivalent to Fibonnaci.

Example: It a known fact that rabbits never die. At time t we have a pairs of breeding rabbits. Each pair produces one pair at ages $2, 3 \ldots$. Therefore, $a_t = a_{t-1} + a_{t-2}$. This is also equivalent to Fibonacci.

Example: Consider n lines in the plane such that every pair intersects, but no more than 3 intersect. Let a_n = regions with n lines. We have that $a_0 = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 = 7$, and $a_4 = 7 + 4 = 11$.

19.2 Solving Recurrence Relations

The condensed method:

1. Figure out the recurrence for the problem and put all terms with a subscript on one side in the form $a_n + \ldots = y$.
2. Is the recurrence homogeneous (as in, is $y = 0$?) ?
3. If yes, then skip this step. If no, then you need to find a particular solution. Try a multiple of y and solve for the constant that makes the solution work with the boundary conditions. Call the solution Z.
4. Make the recurrence into a quadratic equation by using its characteristic equation. Find its roots. Call them X and Y. Now you know that the recurrence has solution $AX^n + BY^n + Z$.
5. Solve for A and B based on the boundary conditions.

Example: Solve the recurrence $a_{n+2} - 4a_{n+1} + 3a_n = 4^n$ where $a_0 = 10/3$ and $a_1 = 19/3$.

Because this recurrence is not homogenous, we need to find a particular solution. This particular solution has to have a factor of 4^n so we try $k4^n$. Substituting we get, $k4^{n+2} - 4k4^{n+1} + 3k4^n = 4^n$. If we solve for k we get $16k - 16k + 3k = 0$ and $k = 1/3$. We now know that the particular solution is $\frac{1}{3}4^k$.

The characteristic equation of this recurrence is of the form $\alpha^2 - 4\alpha + 3 = 0$. The solutions are $\alpha = 1$ and $\alpha = 3$. This means that the recurrence has a general solution of $C + D3^n + \frac{1}{3}4^n$. Based on the fact that $a_0 = C + D = \frac{10}{3}$ and $a_1 = C + 3D = \frac{19}{3}$ we get that $D = 1$ and $C = 2$. This means that the general solution is $2 + 3^n + \frac{1}{3}4^n$.